QUASI-DERIVATIONS OF LOW-DIMENSIONAL LEIBNIZ ALGEBRAS AND THEIR PROPERTIES

Musayev Sardor Xabibulla o'g'li
UNIVERSITY OF SCIENCE AND TECHNOLOGIES
Teacher of the "Exact Sciences" department
sardormusayev1999@gmail.com
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ABSTRACT

The article presents the results obtained about quasi-derivations of small-dimension Leibniz algebras and their properties.

Key words: Leibniz algebra, derivation, quasi-derivation, generalized derivation, centroids and quasi-centroids.

INTRODUCTION

Currently, the class of Leibniz algebras, which is a generalization of Lie algebras, is being intensively studied. It should be noted that algebras satisfying the Leibniz theorem were first introduced in 1965 in the work of A. Bloch under the name of D-algebras. However, not much attention was paid to the study of D-algebras, only J.L. Only after the works of Lode and T. Pirashvili, Leibniz algebras began to be intensively studied, and up to now a number of articles devoted to these algebras have been published. Leibniz algebras were developed by the French mathematician J.L. in the 90s of the last century. This by Lode

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

It was included in the science as an algebra characterized by Leibniz's equation. Since 1998, the structural theory of Leibniz algebra has been developed by Sh.A. Ayupov and B.A. The Omirovs began to learn. The larger the size of the algebra, the more difficult it is to describe. Ayupov Sh.A., Omirov B.A., Rakhimov I.S., Riksiboev I.M., Khudoyberdiyev A.Kh. with nilpotent Leibniz algebras. and others were engaged. Since the study of nilpotent Lie algebras of large dimensions is also complicated, nilpotent algebras are divided into several classes. For example, zero filiform, filiform, quasi filiform and other classes.

In recent years, a number of operators considered as differentiations of non-associative algebras and generalizations of differentiations have been widely studied. In particular, the concepts of quasi-differentiations were studied not only for operator algebras but also for Lie and Leibniz algebras. This article explores the concept of quasi-differentiations of small-scale Leibniz algebras. Quasi-differentiations of small-scale Leibniz algebras and their properties are defined.

1. PRELIMINARIES

This section is devoted to recalling some basic notions and concepts used through the work.

Definition 1. Let $(L,[-,-])$ be an algebra over F. L is called Leibniz algebra, if is satisfies for all x, y and z:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

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Definition 2. A linear map $D: L \to L$ is said to be a derivation of L if it satisfies:

$$D([x,y]) = [D(x),y] + [x,D(y)],$$

for all $x, y \in L$.

We denote the set of all derivations of L by $Der(L)$, and then, $Der(L)$ provided with the commutator is a subalgebra of $End(L)$ and is called the derivation algebra of L.

Definition 3. $D \in End(L)$ is said to be a generalized derivation of L, if there exist $\exists D', D'' \in End(L)$ such that:

$$[D(x),y] + [x,D'(y)] = D''([x,y])$$

for all $x, y \in L$.

Definition 4. $D \in End(L)$ is said to be a quasiderivation of L, if there exists $\exists D' \in End(L)$, such that:

$$D(x),y] + [x,D(y)] = D'([x,y])$$

for all $x, y \in L$.

Denote by $GDer(L)$ and $QDer(L)$ the sets of generalized derivations and quasiderivations, respectively.

Definition 5. If $C(L) = \{D \in End(L)\} \mid [D(x),y] = [x,D(y)] = D([x,y]\}$, for all $x, y \in L$, then $C(L)$ is called the centroid of L.

Definition 6. If $QC(L) = \{D \in End(L)\} \mid [D(x),y] = [x,D(y)]\}$, for all $x, y \in L$, then $QC(L)$ is called the quasicentroid of L.

Definition 7. If $ZDer(L) = \{D \in End(L) \mid [D(x),y] = [x,D(y)] = D([x,y]) = 0\}$, for all $x, y \in L$, then $ZDer(L)$ is called the central derivation of L.

It is easy to verify that

$$ZDer(L) \subseteq Der(L) \subseteq QDer(L) \subseteq GDer(L) \subseteq End(L)$$
$$C(L) \subseteq QC(L) \subseteq QDer(L)$$

Theorem 1. If L is a Leibniz algebra, then:

(1) $[Der(L), C(L)] \subseteq C(L)$

(2) $[QDer(L), QC(L)] \subseteq QC(L)$

(3) $D(Der(L)) \subseteq Der(L), \forall D \in C(L)$

(4) $C(L) \subseteq QDer(L)$

(5) $[QC(L), QC(L)] \subseteq QDer(L)$

(6) $QDer(L) + QC(L) \subseteq GDer(L)$.

Here, the commutator $[-,-]$ is defined as follows: $[D_1, D_2] = D_1 D_2 - D_2 D_1$.

Proposition 1. Let L be a Leibniz algebra over F, with the operation $D_1 \bullet D_2 = D_1 D_2 + D_2 D_1, \forall D_1, D_2 \in End(L)$. Then, $(End(L), \bullet)$ is a Jordan algebra.

QUASI-DERIVATIONS OF TWO-DIMENSIONAL LEIBNIZ ALGEBRAS:

It is known that any two-dimensional Leibniz algebra is isomorphic to one of the following non-isomorphic Leibniz algebras:

$L_1: [e_1, e_1] = e_2$

$L_2: [e_1, e_2] = -[e_2, e_1] = e_2$

$L_3: [e_1, e_2] = [e_2, e_2] = e_1$

2. MAIN RESULTS:

We define derivations, quasi-derivations, centroids, quasi-centroids and generalization derivations of these three different 2-dimensional algebras:

Theorem 2. Derivations of algebra $L_1: [e_1, e_1] = e_2$ has the form:

$L_1: [e_1, e_1] = e_2$				
$Der(L_1)$	$QDer(L_1)$	$GDer(L_1)$	$C(L_1)$	$QC(L_1)$
$\begin{pmatrix} d_{11} & d_{12} \\ 0 & 2d_{11} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{11} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}$

Theorem 3. Derivations of algebra $L_2: [e_1, e_2] = -[e_2, e_1] = e_2$ has the form:

$L_2: [e_1, e_2] = -[e_2, e_1] = e_2$				
$Der(L_2)$	$QDer(L_2)$	$GDer(L_2)$	$C(L_2)$	$QC(L_2)$
$\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$

Theorem 4. Derivations of algebra $L_3: [e_1, e_2] = [e_2, e_2] = e_1$ has the form:

$L_3: [e_1, e_2] = [e_2, e_2] = e_1$				
$Der(L_3)$	$QDer(L_3)$	$GDer(L_3)$	$C(L_3)$	$QC(L_3)$
$\begin{pmatrix} d_{11} & 0 \\ d_{11} & 0 \end{pmatrix}$	$\begin{pmatrix} d_{21} + d_{22} & 0 \\ d_{21} & d_{22} \end{pmatrix}$	$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$

QUASI-DERIVATIONS OF THREE-DIMENSIONAL NILPOTENT LEIBNIZ ALGEBRAS:

We are given the following three-dimensional nilpotent Leibniz algebras:
λ_1: $abelian$;
λ_2: $[e_1, e_1] = e_2$;
λ_3: $[e_2, e_3] = e_1$, $[e_3, e_2] = -e_1$;
λ_4: $[e_2, e_1] = e_3$, $[e_1, e_2] = \alpha e_3$, $\alpha \neq \alpha^{-1} (\alpha \in C)$;
λ_5: $[e_1, e_1] = e_3$, $[e_2, e_1] = e_3$, $[e_1, e_2] = -e_3$
λ_6: $[e_1, e_1] = e_2$, $[e_2, e_1] = e_3$.
We find a set of quasi-derivations for these algebras.

Proposition 2. The general representation of the matrix of the space of all derivations of λ_1: *abelian* algebra is as follows:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

Theorem 5. The general representation of the matrix of the space of all quasi-derivations of λ_2: $[e_1, e_1] = e_2$ algebra is as follows:

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$$QDer(\lambda_2) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & \gamma_2 & \gamma_3 \end{pmatrix}$$

We present the remaining derivations of the λ_2 algebra without proof in the following table:

$\lambda_2: [e_1, e_1] = e_2$			
$Der(\lambda_2)$	$GDer(\lambda_2)$	$C(\lambda_2)$	$QC(\lambda_2)$
$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 2\alpha_1 & 0 \\ 0 & \gamma_2 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & \gamma_2 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \alpha_1 & 0 \\ 0 & \gamma_2 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & \gamma_2 & \gamma_3 \end{pmatrix}$

Theorem 6. The general representation of the matrix of the space of all quasi-derivations of λ_3: $[e_2, e_3] = e_1,\ [e_3, e_2] = -e_1$ algebra is as follows:

$$QDer(\lambda_3) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

$\lambda_3: [e_2, e_3] = e_1, [e_3, e_2] = -e_1$			
$Der(\lambda_3)$	$GDer(\lambda_3)$	$C(\lambda_3)$	$QC(\lambda_3)$
$\begin{pmatrix} \beta_2+\gamma_3 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 \\ \gamma_1 & 0 & \alpha_1 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & 0 & \gamma_3 \end{pmatrix}$

Theorem 7. The general representation of the matrix of the space of all quasi-derivations of λ_4: $[e_2, e_1] = e_3,\ [e_1, e_2] = \alpha e_3,\ \alpha \neq \alpha^{-1}, (\alpha \in C)$ algebra is as follows:

$$QDer(\lambda_4) = \begin{pmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix}$$

$\lambda_4:\ [e_2, e_1] = e_3,\ [e_1, e_2] = \alpha e_3,\ \alpha \neq \alpha^{-1}, (\alpha \in C)$			
$Der(\lambda_4)$	$GDer(\lambda_4)$	$C(\lambda_4)$	$QC(\lambda_4)$
$\begin{pmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & 0 & \alpha_1+\beta_2 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \alpha_1 & \beta_3 \\ 0 & 0 & \alpha_1 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix}$

Theorem 8. The general representation of the matrix of the space of all quasi-derivations of λ_5: $[e_1,e_1]=e_3,\ [e_2,e_1]=e_3,\ [e_1,e_2]=-e_3$ algebra is as follows:

$$QDer(\lambda_5)=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 0 & \alpha_1 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix}$$

$\lambda_5: [e_1,e_1]=e_3, [e_2,e_1]=e_3, [e_1,e_2]=-e_3$			
$Der(\lambda_5)$	$GDer(\lambda_5)$	$C(\lambda_5)$	$QC(\lambda_5)$
$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 0 & \alpha_1 & \beta_3\\ 0 & 0 & 2\alpha_1\end{pmatrix}$	$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 0 & \beta_2 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix}$	$\begin{pmatrix}\alpha_1 & 0 & \alpha_3\\ 0 & \alpha_1 & \beta_3\\ 0 & 0 & \alpha_1\end{pmatrix}$	$\begin{pmatrix}\alpha_1 & 0 & \alpha_3\\ 0 & \alpha_1 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix}$

Theorem 9. The general representation of the matrix of the space of all quasi-derivations of λ_6: $[e_1,e_1]=e_2,\ [e_2,e_1]=e_3$ algebra is as follows:

$$QDer(\lambda_6)=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 0 & \beta_2 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix}$$

$\lambda_6: [e_1,e_1]=e_2, [e_2,e_1]=e_3$			
$Der(\lambda_6)$	$GDer(\lambda_6)$	$C(\lambda_6)$	$QC(\lambda_6)$
$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 0 & 2\alpha_1 & \alpha_2\\ 0 & 0 & 3\alpha_1\end{pmatrix}$	$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix}$	$\begin{pmatrix}\alpha_1 & 0 & \alpha_3\\ 0 & \alpha_1 & 0\\ 0 & 0 & \alpha_1\end{pmatrix}$	$\begin{pmatrix}\alpha_1 & 0 & \alpha_3\\ 0 & \alpha_1 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix}$

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