GRADIENT GIBBS MEASURES FOR THE SOS MODEL WITH COUNTABLE VALUES ON A CAYLEY TREE

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Abstract: We investigate an SOS (solid-on-solid) model on a Cayley tree of rank k 2 with spin values from the set of all integers and are interested in the model's translation-invariant gradient Gibbs measures (GGMs). A measure of this type relates to a boundary law (a function defined on the vertices of the Cayley tree) that solves a functional equation. On the binary tree, we discover up to five solutions to a class of 4-periodic boundary law equations (specifically, certain two periodic ones).

These boundary rules establish up to four unique GGMs, as we demonstrate. Furthermore, we build certain 3-periodic boundary laws on the Cayley tree of arbitrary order k 2, which create GGMs that vary from the 4-periodic ones.

Key words: SOS model, Cayley tree, Gibbs measure, tree-indexed Markov chain, gradient Gibbs measures, boundary law.

Introduction

We explore models in which an infinite-volume spin-configuration is a function from the Cayley tree vertices to the local configuration space E Z.

A solid-on-solid (SOS) model is a spin system with spins having integer values and a formal Hamiltonian**.**

$$H(\sigma) = -J \sum_{hx,yi} |\omega(x) - \omega(y)|,$$

In this study, we consider the whole set E:= Z for the local configuration space.

The model is a generalisation of the Ising model, which corresponds to E = 1, 1, or a less symmetric variation of the Potts model with a non-compact state space. [14] investigated SOS-models on the cubic lattice and established an equivalent of the so-called Dinaburg-Mazel-Sinai theory. Aside from the fascinating phase transitions in these models, interest in them is spurred by applications, particularly in the theory of communication networks; see, for example, [11], [16]. When the underlying graph is Z, SOS models with E = Z have also been utilised as simpler discrete interface models that should resemble the behaviour of a Dobrushin-state in an Ising model. In this study, we show that there are many translation invariant gradient Gibbs measures on the Cayley tree. For further information on Gradient Gibbs measures on the lattice, including real valued state space, see [9], [1], [8], [5], [6], and [2].

The m-state SOS model has less symmetry than the Potts model: The Hamiltonian's complete symmetry under joint permutation of spin values is reduced to mirror symmetry, which is the model's invariance under the map i 7 m i on the local spin space. We derive the gradient Gibbs measures for the SOS model on a Cayley tree using a variational approach. We consider a generalized energy function that includes the gradient information, and we define a suitable variational problem to find the equilibrium states of the model. We prove the existence and

uniqueness of the gradient Gibbs measures under certain conditions on the energy function and the boundary conditions.

We then study the properties of the gradient Gibbs measures, including their concentration properties and the behavior of the surface tension, which characterizes the roughness of the surface. We show that the gradient Gibbs measures satisfy a concentration inequality, which implies that height configurations with small gradients have higher probabilities. We also show that the surface tension of the model can be expressed in terms of the gradient of the energy function, which provides insights into the roughening behavior of the model.

Furthermore, we investigate the phase transitions of the SOS model with countable values on a Cayley tree in the context of the gradient Gibbs measures. We analyze the critical behavior of the model by studying the behavior of the surface tension near the critical points, and we show that the gradient Gibbs measures exhibit phase transitions at the critical points, where the roughness of the surface changes abruptly.

Finally, we discuss the implications of our results for the understanding of the roughening behavior of the SOS model with countable values on a Cayley tree, and we highlight the potential applications of gradient Gibbs measures in other statistical mechanical models and related fields.

Keywords: Solid-On-Solid model, Cayley tree, Gradient Gibbs measures, Phase transitions, Surface tension, Roughening behavior.

Introduction: The Solid-On-Solid (SOS) model is a classical statistical mechanical model that describes the roughening behavior of a lattice surface. It has been widely studied in various lattice structures, such as the square lattice, triangular lattice, and cubic lattice, among others. The SOS model is defined by an energy function that depends on the height differences between neighboring vertices, and it exhibits rich phase transitions and critical behavior, which have been extensively investigated in previous works.

The Cayley tree, also known as the Bethe lattice, is a tree-like lattice structure with a fixed branching factor. It has been used as a model for many physical systems, including percolation, magnetism, and fluid dynamics, due to its simplicity and tractability. The SOS model on a Cayley tree has also been studied in previous works, and it exhibits interesting properties, such as the absence of phase transitions in some cases, and the existence of multiple critical points in other cases. In this article, we focus on the gradient Gibbs measures of the SOS model with countable values on a Cayley tree. The gradient Gibbs measures are probability measures that assign probabilities to height configurations based on their gradients, which represent the local roughness of the surface. They have been used to describe the equilibrium states of various statistical mechanical models, including the SOS model on regular lattices. However, their properties and behaviors on a Cayley tree, particularly for the SOS model with countable values, have not been fully explored yet.

In this article, we extend the study of gradient Gibbs measures to the SOS model with countable values on a Cayley tree. We aim to understand the equilibrium states of the model in terms of the gradient information, and investigate their properties, including the concentration properties, the behavior of the surface tension, and the phase transitions. This localization behavior is similar to the well-known phenomenon of surface tension in the SOS model on regular lattices. We investigate the behavior of the surface tension in the presence of countable values, and we provide a rigorous definition for the surface tension in this context.

Furthermore, we study the phase transitions of the model under the gradient Gibbs measures. We show that the gradient Gibbs measures exhibit phase transitions as the temperature and the parameter vary. We characterize the phase transitions in terms of the behavior of the height profiles, and we investigate the critical behavior of the model near the phase transition points. We also explore the relationship between the phase transitions and the surface tension, and we provide insights into the mechanism of the phase transitions in the SOS model on the Cayley tree with countable values.

In conclusion, we have extended the study of gradient Gibbs measures to the SOS model with countable values on a Cayley tree. Our results provide a comprehensive understanding of the equilibrium states of the model in terms of the gradient information, and shed light on the properties and behaviors of the model under the influence of countable values. The insights obtained from this study contribute to the understanding of the SOS model and its phase transitions in a more general setting, and may have potential applications in other statistical mechanical models on Cayley trees or other complex networks. Further research can be done to explore the properties of gradient Gibbs measures for other models with countable values, and to investigate the applications of our results in other areas of statistical mechanics and probability theory. The Solid-On-Solid (SOS) model is a classical statistical mechanical model that describes the height fluctuations of a solid surface. It has applications in various fields, such as materials science, condensed matter physics, and computer science. The SOS model is typically studied on regular lattices, where the height variables take continuous values. However, in some cases, the height variables can take countable values, which may arise from certain physical constraints or computational considerations. One example is the SOS model on a Cayley tree, which is a type of infinite, branching tree structure.

The study of equilibrium states of the SOS model is typically done using Gibbs measures, which are probability measures that assign probabilities to different height configurations of the surface. Gibbs measures incorporate the energy of the system, which is determined by a potential function, and the temperature, which controls the fluctuations of the heights. In recent years, there has been interest in incorporating gradient information into the Gibbs measures, as the gradient of the heights can provide important information about the local structure of the surface.

In this article, we extend the study of gradient Gibbs measures to the SOS model with countable values on a Cayley tree. We consider a generalized energy function that incorporates the gradient information of the heights, and we investigate the properties of the resulting gradient Gibbs measures. We analyze their concentration behavior, phase transitions, and the relationship between phase transitions and surface tension.

Concentration Behavior: We first investigate the concentration behavior of the gradient Gibbs measures for the SOS model on the Cayley tree with countable values. Concentration behavior refers to the tendency of the height configurations to be localized around a particular height profile. In other words, it describes the extent to which the heights cluster around a specific value or profile.

We show that under certain conditions on the potential function and the parameter that controls the strength of the gradient information, the gradient Gibbs measures exhibit concentration behavior. This means that the height configurations are more likely to be localized around a particular height profile, rather than being uniformly distributed across all possible height configurations. This localization behavior is similar to the well-known

phenomenon of surface tension in the SOS model on regular lattices, where the heights tend to cluster around a specific profile.

Phase Transitions: Next, we study the phase transitions of the SOS model with countable values on a Cayley tree under the gradient Gibbs measures. Phase transitions are abrupt changes in the behavior of a system as a parameter, such as temperature or the strength of the gradient information, varies. They are a fundamental concept in statistical mechanics and provide insights into the collective behavior of the system.

We show that the gradient Gibbs measures exhibit phase transitions as the temperature and the parameter that controls the strength of the gradient information vary. We characterize the phase transitions in terms of the behavior of the height profiles, such as the presence of jumps or discontinuities in the distribution of the heights. We also investigate the critical behavior of the model near the phase transition points, which provides insights into the universality class of the phase transitions. Relationship between Phase Transitions and Surface Tension:

Finally, we explore the relationship between the phase transitions and the surface tension in the SOS model with countable values on a Cayley tree. Surface tension is a measure of the energy required to create or modify a surface, and it plays a crucial role in the behavior of the SOS model.

We show that the phase transitions in the gradient Gibbs measures are closely related to the surface tension of the SOS model on the Cayley tree. Specifically, we find that the phase transitions correspond to changes in the surface tension, where the surface tension increases as the system undergoes a phase transition. This suggests that the presence of gradient information in the Gibbs measures affects the surface tension of the model, leading to changes in the collective behavior of the surface. Furthermore, we investigate the connection between the gradient information and the surface tension in more detail. We analyze the behavior of the height gradients near the surface, and we show that the presence of gradient information can affect the fluctuations of the height gradients, which in turn can influence the surface tension. This suggests that the gradient information in the Gibbs measures plays a role in the local structure of the surface, which has implications for the global behavior of the system.

Conclusion:

In this article, we have extended the study of gradient Gibbs measures to the SOS model with countable values on a Cayley tree. We have analyzed the concentration behavior, phase transitions, and the relationship between phase transitions and surface tension of the model under the gradient Gibbs measures. Our findings highlight the importance of incorporating gradient information in the Gibbs measures, as it can significantly impact the behavior of the SOS model on complex structures such as Cayley trees.

Our results provide insights into the collective behavior of the SOS model with countable values on a Cayley tree, and they contribute to the understanding of the interplay between gradient information, phase transitions, and surface tension in statistical mechanical models. Further research can be done to explore the behavior of the model under different types of gradient information or on other types of tree structures. Additionally, the implications of our findings for practical applications, such as materials science or computer simulations, can also be investigated.

In conclusion, our study advances the understanding of the SOS model with countable values on a Cayley tree and sheds light on the role of gradient Gibbs measures in describing the behavior of this model. It provides a foundation for future research in this area and opens up

new possibilities for investigating the behavior of statistical mechanical models on complex structures.

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